Algebra 2

An Incremental Development

Third Edition

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SAXON PUBLISHERS, INC.

Contents

Preface

This is the third edition of the second book in an integrated three-book series designed to prepare students for calculus. In this book we continue the study of topics from algebra and geometry and begin our study of trigonometry. Mathematics is an abstract study of the behavior and interrelationships of numbers. In *Algebra 1* we found that algebra is not difficult—it is just different. Concepts that were confusing when first encountered became familiar concepts after they had been practiced for a period of weeks or months—until finally they were understood. Then further study of the same concepts caused additional understanding as totally unexpected ramifications appeared. And, as we mastered these new abstractions, our understanding of seemingly unrelated concepts became clearer.

Thus, mathematics does not consist of unconnected topics that can be filed in separate compartments, studied once, mastered, and then neglected. Mathematics is like a big ball made of pieces of string that have been tied together. Many pieces touch directly, but the other pieces are all an integral part of the ball, and all must be rolled along together if understanding is to be achieved.

A total assimilation of the fundamentals of mathematics is the key that will unlock the doors of higher mathematics and the doors to chemistry, physics, engineering, and other mathematically based disciplines. In addition, it will also unlock the doors to the understanding of psychology, sociology, and other nonmathematical disciplines in which research depends heavily on mathematical statistics. Thus, we see that mathematical ability is necessary in almost any field of endeavor.

One must be able to apply the fundamental concepts of mathematics automatically if these fundamentals are to be useful. There is insufficient time to relearn basics every time a basic principle must be applied, and familiarity or a slight acquaintance with a basic principle does not suffice for its use.

Thus, in this book we go back to the beginning—to signed numbers—and then quickly review all of the topics of *Algebra 1* and practice these topics as we weave in more advanced concepts. We will also practice the skills that are necessary to apply the concepts. The applicability of some of these skills, such as completing the square, deriving the quadratic formula, simplification of radicals, and complex numbers, might not be apparent at this time, but the benefits of having mastered these skills will become evident as your education continues.

We will continue our study of geometry in this book. Lessons on geometry appear at regular intervals, and one or two geometry problems appear in every homework problem set. We begin our study of trigonometry in Lesson 43 when we introduce the fundamental trigonometric ratios—the sine, cosine, and tangent. We will practice the use of these ratios in every problem set for the rest of the book. The long-term practice of the fundamental concepts of algebra, geometry, and trigonometry will make these concepts familiar concepts and will enable an in-depth understanding of their use in the next book in this series, a precalculus book entitled *Advanced Mathematics*.

Problems have been selected in various skill areas, and these problems will be practiced again and again in the problem sets. It is wise to strive for speed and accuracy when working these review problems. If you feel that you have mastered a type of problem, don't skip it when it appears again. If you have really mastered the concept, the problem should not be troublesome; you should be able to do the problem quickly and accurately. If you have not mastered the concept, you need the practice that working the problem will provide. You must work every problem in every problem set to get the full benefit of the structure of this book. Master musicians practice fundamental musical skills every day. All experts practice fundamentals as often as possible. To attain and maintain proficiency in mathematics, it is necessary to practice fundamental mathematical skills constantly as new concepts are being investigated. And, as in the last book, you are encouraged to be diligent and to work at developing defense mechanisms whose use will protect you against every human's seemingly uncanny ability to invent ways to make mistakes.

One last word. There is no requirement that you like mathematics. I am not especially fond of mathematics—and I wrote the book—but I do love the ability to pass through doors that knowledge of mathematics has unlocked for me. I did not know what was behind the doors when I began. Some things I found there were not appealing, while others were fascinating. For example, I enjoyed being an air force test pilot. A degree in engineering was a requirement to be admitted to test-pilot school. My knowledge of mathematics enabled me to obtain this degree. At the time I began my study of mathematics, I had no idea that I would want to be a test pilot or would ever need to use mathematics in any way.

I again thank Frank Wang for his valuable help in getting the first edition of this book finalized and publisher Bob Worth for his help in getting the first edition published.

John Saxon

Norman, Oklahoma

LESSON A *Geometry Review • Angles • Review of Absolute Value • Properties and Definitions*

A.A geometry review

Some fundamental mathematical terms are impossible to define exactly. We call these terms **primitive terms** or **undefined terms.** We define these terms as best we can and then use them to define other terms. The words **point, curve, line,** and **plane** are primitive terms.

A **point** is a location. When we put a dot on a piece of paper to mark a location, the dot is not the point, because a mathematical point has no size and the dot does have size. We say that the dot is the **graph** of the mathematical point and marks the location of the point. A **curve** is an unbroken connection of points. Since points have no size, they cannot really be connected. Thus, we prefer to say that a curve defines the path traveled by a moving point. We can use a pencil to graph a curve. These figures are curves.

A mathematical **line** is a straight curve that has no ends. **Only one mathematical line can be drawn that passes through two designated points.** Since a line defines the path of a moving point that has no width, a line has no width. The pencil line that we draw marks the location of the mathematical line. When we use a pencil to draw the graph of a mathematical line, we often put arrowheads on the ends of the pencil line to emphasize that the mathematical line has no ends.

We can name a line by naming any two points on the line in any order. The line above can be called line *AB,* line *BA,* line *AC,* line *CA,* line *BC,* or line *CB.* Instead of writing the word *line,* we can put a bar with two arrowheads above the letters, as we show here.

$$\overleftrightarrow{AB} \qquad \overleftrightarrow{BA} \qquad \overleftrightarrow{AC} \qquad \overleftrightarrow{CA} \qquad \overleftrightarrow{BC} \qquad \overleftrightarrow{CB}$$

These notations are read as "line *AB,*" "line *BA,*" etc. We remember that a part of a line is called a **line segment** or just a **segment.** A segment has two endpoints. A segment can be named by naming the two endpoints in any order. The following segment can be called segment *AB* or segment *BA.*

Instead of writing the word *segment,* we can draw a bar with no arrowheads above the letters. Segment *AB* and segment *BA* can be written as

$$\overline{AB} \qquad \text{and} \qquad \overline{BA}$$

If we write the letters without using the bar, we are designating the length of the segment. If segment AB has a length of 2 centimeters, we could write either

$$AB = 2 \text{ cm} \qquad \text{or} \qquad BA = 2 \text{ cm}$$

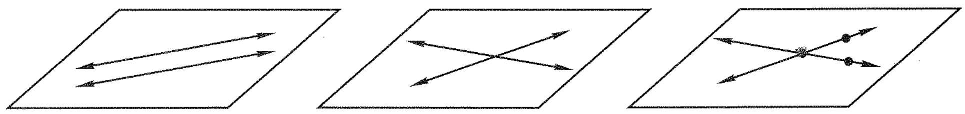
A **ray** is sometimes called a **half line.** A ray has one endpoint, the beginning point, called the **origin.** The ray shown here begins at point A, goes through points B and C, and continues without end.

When we name a ray, we must name the origin first and then name any other point on the ray. We can name a ray by using a line segment with one arrowhead. The ray shown above can be named by writing either

$$\overrightarrow{AB} \qquad \text{or} \qquad \overrightarrow{AC}$$

These notations are read by saying "ray AB" and "ray AC."

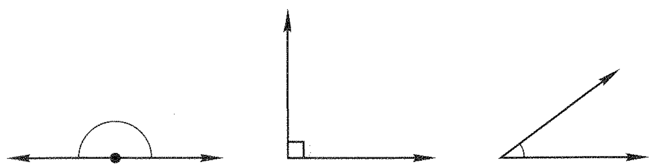
A **plane** is a flat surface that has no boundaries and no thickness. Two lines in the same plane either **intersect** (cross) or do not intersect. Lines in the same plane that do not intersect are called **parallel lines.** All points that lie on either of two intersecting lines are in the plane that contains the lines. We say that these intersecting lines determine the plane. Since three points that are not on the same line determine two intersecting lines, we see that three points that are not on the same line also determine a plane.



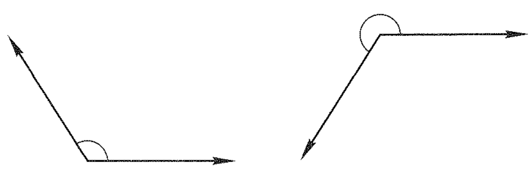
Parallel lines — Intersecting lines — Three points

A.B angles

The word **angle** comes from the Latin word *angulum,* meaning "corner." An angle is formed by two rays that have a common endpoint. If the rays point in opposite directions, we say that the angle formed is a **straight angle.** If the rays make a square corner, we say that the rays are **perpendicular** and that the angle formed is a **right angle.** We often use a small square, as in the following figure, to designate a right angle. If the angle is smaller than a right angle, it is an **acute angle.** An angle greater than a right angle but less than a straight angle is called an **obtuse angle.** An angle greater than a straight angle but less than two straight angles is called a **reflex angle.**



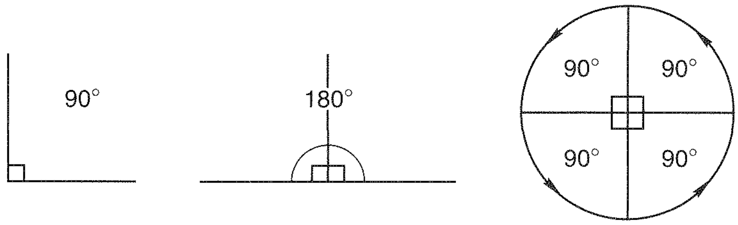
Straight angle — Right angle — Acute angle



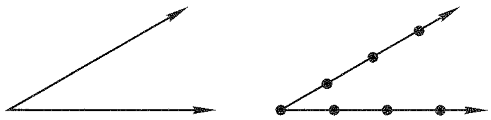
Obtuse angle — Reflex angle

If a right angle is divided into 90 parts, we say that each part has a measure of 1 degree. Thus, a right angle is a **90-degree angle.** Two right angles make a straight angle, so a

straight angle is a 180-degree angle. Four right angles form a **360-degree angle.** Thus, the measure of a circle is 360 degrees. We use a small raised circle to denote degrees. Thus, we can write 90 degrees, 180 degrees, and 360 degrees as 90°, 180°, and 360°.

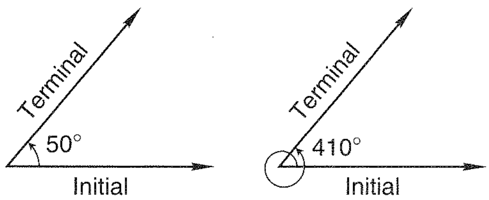


European authors tend to define an angle to be the **opening** between two rays. Authors of U.S. geometry books tend to define the angle to be the **set of points** determined by the two rays.

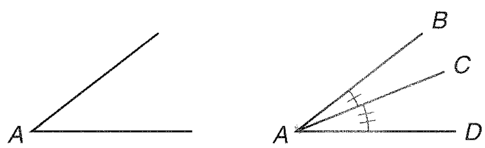


Opening Set of points

Authors of trigonometry books prefer to define an angle to be a **rotation** of a ray about its endpoint from an **initial position** to a final position called the **terminal position.** We see that the rotation definition permits us to distinguish between a 50° angle and a 410° angle even though the initial and terminal positions are the same.

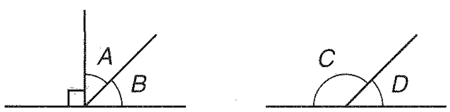


Some angles can be named by using a single letter preceded by the symbol ∠. The notation ∠*A* is read as "angle *A*." Some angles require that we use three letters to name the angle. The notation ∠*BAD* is read as "angle *BAD*." When we use three letters, the middle letter names the **vertex** of the angle, which is the point where the two rays of the angle intersect. The other two letters name a point on one ray and a point on the other ray.



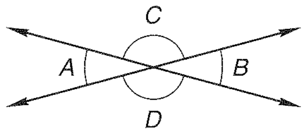
The angle on the left is ∠*A*. The figure on the right has three angles. The big angle is ∠*BAD*. Angle *BAC* and angle *CAD* are called **adjacent angles** because they have the same vertex, share a common side, and do not overlap (i.e., do not have any common interior points).

If the sum of the measures of two angles is 90°, the angles are called **complementary angles.** If the sum of the measures of two angles is 180°, the angles are called **supplementary angles.**



Complementary angles Supplementary angles

In the figures in this book, lines that appear to be straight are straight. Two intersecting lines (all lines are straight lines) form four angles. The angles that are opposite each other are called **vertical angles.** Vertical angles are equal angles.



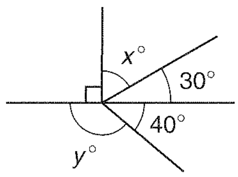
In this figure, angle A has the same measure as angle B, and angle C has the same measure as angle D.

It is important to remember that only numbers can be equal. **If we say that two angles are equal, we mean that the number that describes the measure of one angle is equal to the number that describes the measure of the other angle. If we say that two line segments are equal, we mean that the numbers that describe the lengths of the segments are equal.** Both of the following notations tell us that the measure of angle A equals the measure of angle B.

$$\angle A = \angle B \qquad m\angle A = m\angle B$$

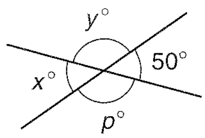
Because excessive attention to the difference between *equal* and *equal measure* tends to be counterproductive, in this book we will sometimes say that angles are equal or that line segments are equal, because this phrasing is easily understood. **However, we must remember that when we use the words *equal angles* or *equal segments,* we are describing angles whose measures are equal and segments whose lengths are equal.**

example A.1 Find x and y.



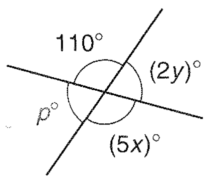
solution The 30° angle and angle x form a right angle, so x equals **60.** Thus, angle x and the 30° angle are **complementary** angles. The 40° angle and angle y form a straight angle. Straight angles are 180° angles, so y equals **140.** Thus, angle y and the 40° angle are **supplementary** angles.

example A.2 Find x, y, and p.



solution Angle y and the 50° angle form a 180° angle. Thus, y equals **130.** Because vertical angles are equal angles, x equals **50** and p equals **130.**

example A.3 Find x, y, and p.



solution This problem allows us to use the fact that if two angles form a straight angle, the sum of their measures is 180°. We see that angle $2y$ and 110° form a straight angle. Also, $5x$ must equal 110° because vertical angles are equal.

STRAIGHT ANGLE	VERTICAL ANGLE
$2y + 110 = 180$	$5x = 110$
$2y = 70$	$\mathbf{x = 22}$
$\mathbf{y = 35}$	

Since y is 35, $2y$ is 70. Thus, $\mathbf{p = 70}$ because vertical angles are equal.

A.C review of absolute value

A number is an idea. A numerical expression is often called a numeral and is a single symbol or a collection of symbols that designates a particular number. We say that the number designated is the value of the expression. All of the following numerical expressions designate the number positive three, and we say that each of these expressions has a value of positive three.

$$3 \qquad \frac{7+8}{5} \qquad 2+1 \qquad \frac{12}{4} \qquad \frac{75}{25} \qquad \frac{16}{2}-5$$

We have agreed that a positive number can be designated by a numeral preceded by a plus sign or by a numeral without a sign. Thus, we can designate positive three by writing either

$$+3 \qquad \text{or} \qquad 3$$

The number zero is neither positive nor negative and can be designated with the single symbol

$$0$$

Every other real number is either positive or negative and can be thought of as having two qualities or parts. One of the parts is designated by the plus sign or the minus sign. The other part is designated by the numerical part of the numeral. The two numerals

$$+3 \qquad \text{and} \qquad -3$$

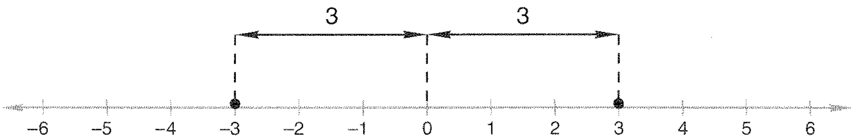
designate a positive number and a negative number. The signs of the numerals are different, but the numerical part of each is

$$3$$

We say that this part of the numeral designates the **absolute value** of the number. It is difficult to find a definition of absolute value that is acceptable to everyone. Some people object to saying that the absolute value is the same thing as the "bigness" of a number, because "bigness" might be confused with the concept of "greater than," which is used to order numbers. Some explain absolute value by saying that all nonzero real numbers can be paired, each with its opposite, and that the absolute value of either is the positive member of the pair. Thus,

$$+3 \qquad \text{and} \qquad -3$$

are a pair of opposites, and both have an absolute value of 3. Other people prefer to define the absolute value of a number as the number that describes the distance of the graph of the number from the origin. If we use this definition, we see that the graphs of +3 and −3 are both 3 units from the origin, and thus both numbers have an absolute value of 3.



Some people feel that words should not be used to define absolute value, because absolute value can be defined exactly by using only symbols and using two vertical lines to indicate absolute value. This definition is in three parts. Unfortunately, the third part can be confusing.

$$\text{(a) If } x > 0,\ |x| = x$$
$$\text{(b) If } x = 0,\ |x| = x$$
$$\text{(c) If } x < 0,\ |x| = -x$$

Part (c) does not say that the absolute value of x is a negative number. It says that if x is a negative number (all numbers less than zero are negative), the absolute value of x is the opposite of x. Since −15 is a negative number, its absolute value is its opposite, which is +15.

$$|-15| = -(-15) = 15$$

In the same way, if we designate the absolute value of an algebraic expression such as

$$|x + 2|$$

and x has a value such that $x + 2$ is a negative number, then the absolute value of the expression will be the negative of the expression. If $x + 2 < 0$,

$$|x + 2| = -(x + 2)$$

To demonstrate, we give x a value of –5, and then we will have

$$|-5 + 2| = |-3| = -(-3) = +3$$

No matter how we think of absolute value, we must remember that the absolute value of zero is zero and that the absolute value of every other real number is a positive number.

$$|0| = 0 \qquad |-5| = 5 \qquad |5| = 5 \qquad |-2.5| = 2.5$$

In this book we will sometimes use the word *number* when the word *numeral* would be more accurate. We do this because overemphasizing the distinction between the two words can be counterproductive.

example A.4 Simplify: $-|-4| - 2 + |-5|$

solution We will simplify in two steps.

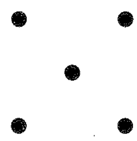
$-4 - 2 + 5$	simplified
$\mathbf{-1}$	added algebraically

A.D properties and definitions

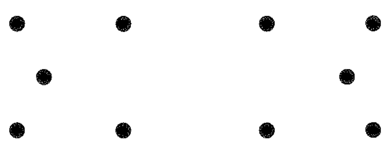
Understanding algebra is easier if we make an effort to remember the difference between properties and definitions. A **property** describes the way something is. We can't change properties. We are stuck with properties because they are what they are. For instance,

$$3 + 2 = 5 \qquad \text{and} \qquad 2 + 3 = 5$$

The order of addition of two real numbers does not change the answer. We can understand this property better if we use dots rather than numerals.



Here we have represented the number 5 with 5 dots. Now, on the left below we separate the dots to show what we mean by $3 + 2$, and on the right we show $2 + 3$. The answer is 5 in both cases because there is a total of five dots regardless of the way in which they are arranged. We call this property the **commutative property of real numbers in addition.**



$3 + 2$ $\qquad$ $2 + 3$

Definitions are different because they are things that we have agreed on. For instance,

$$3^2 \qquad \text{means} \qquad 3 \text{ times } 3$$

It did not have to mean that. We could have used 3^2 to mean "3 times 2," but we did not. We note that the order of operations is also a definition. When we write

$$3 + 4 \cdot 5$$

we could mean to multiply first or to add first. Since we cannot have two different answers to the same problem, it is necessary to agree on the meaning of the notation. We have agreed to do multiplication before algebraic addition, and so this expression represents the number +23.

Also, when we wish to write the negative of 3^2, we write

$$-3^2$$

When we wish to indicate that the quantity –3 is to be squared, we write

$$(-3)^2$$

These are definitions of what we mean when we write

$$-3^2 \qquad \text{and} \qquad (-3)^2$$

and there is nothing to understand. We have defined these notations to have the meanings shown.

The first problem set contains review problems that require us to simplify expressions that contain signed numbers. When these expressions are simplified, try to remember which steps can be justified by properties and which steps can be justified by definitions.

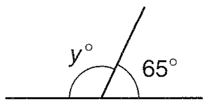
example A.5 Simplify: $(-2)^3 - 2^2 - (-2)^2$

solution First we simplify each expression. Then we work the problem.

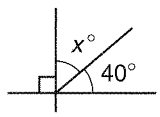
$$-8 - 4 - 4 \qquad \text{simplified}$$
$$\mathbf{-16} \qquad \text{added algebraically}$$

problem set A

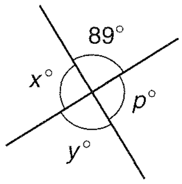
1. Find y.



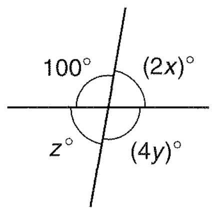
2. Find x.



3. Find x, y, and p.



4. Find x, y, and z.



5. The supplement of an angle is 40°. What is the angle?
6. The complement of an angle is 40°. What is the angle?

The following problems review operations with signed numbers. Remember that $(-2)^2$ means $(-2)(-2)$, which equals +4, and that -2^2 means $-(2)(2)$, which equals –4.

Simplify:

7. $-2 - (-2)$
8. $-3 - [-(-2)]$
9. $-2 - 3(-2 - 2) - 5(-5 + 7)$
10. $-[-2(-5 + 2) - (-2 - 3)]$
11. $-2 + (-2)^3$
12. $-3^2 - 3 - (-3)^2$
13. $-3(-2 - 3 + 6) - [-5(-2) + 3(-2 - 4)]$
14. $-2 - 2^2 - 2^3 - 2^4$
15. $|-2| - |-4 - 2| + |8|$
16. $-|-3(2) - 3| - 2^2$
17. $-2^2 - 2^3 - |-2| - 2$
18. $-3[-1 - 2(-1 - 1)][-3(-2) - 1]$
19. $-3[-3(-4 - 1) - (-3 - 4)]$
20. $-2[(-3 + 1) - (-2 - 2)(-1 + 3)]$
21. $-2[-2(-4) - 2^3](-|2|)$
22. $-8 - 3^2 - (-2)^2 - 3(-2) + 2$
23. $-\{-[-5(-3 + 2)7]\}$
24. $-5 - |-3 - 4| - (3)^2 - 3$
25. $3(-2 + 5) - 2^2(2 - 3) - |-2|$
26. $\dfrac{-5 - (-2) + 8 - 4(5)}{6 - 4(-3)}$
27. $(-2)[|-3 - 4 - 5| - 2^3 - (-1)]$
28. $\dfrac{-3 - (-2) + 9 - (-5)}{7(|-3 + 4|)}$
29. $4(-2)[-(7 - 3)(5 - 2)2]$
30. $4 - (-4) - 5(3 - 1) + 3(4)(-2)^3$